

On the two-dimensional steady Navier-Stokes equations related to flows around a rotating obstacle

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1 Introduction

Let \mathcal{B} be a rigid body immersed in a viscous incompressible fluid that fills the whole space. Assume that the body rotates with a constant angular velocity $a \in \mathbb{R} \setminus \{0\}$ and the exterior of $\mathcal{B}(t)$ is described as $\Omega(t) \subset \mathbb{R}^2$. The time dependent domain $\Omega(t)$ is defined as

$$\begin{aligned}\Omega(t) &= \{y \in \mathbb{R}^2 \mid y = O(at)x, x \in \Omega\}, \\ O(at) &= \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix},\end{aligned}\tag{1}$$

where a given exterior domain $\Omega(0) = \Omega \subset \mathbb{R}^2$ has a smooth boundary $\partial\Omega$. The flow around the rotating body is described by the following Navier-Stokes equations:

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \nabla q = g, & t > 0, y \in \Omega(t), \\ \operatorname{div} v = 0, & t > 0, y \in \Omega(t). \end{cases}\tag{2}$$

Here $v = v(y, t) = (v_1(y, t), v_2(y, t))^\top$ and $q = q(y, t)$ are respectively unknown velocity field and pressure field, and $g = g(y, t) = (g_1(y, t), g_2(y, t))^\top$ is a given external force. We use the standard notation for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} v = \sum_{j=1}^2 \partial_j v_j$, $v \cdot \nabla v = \sum_{j=1}^2 v_j \partial_j v$, while $x^\perp = (-x_2, x_1)^\top$ denotes the vector which is perpendicular to $x = (x_1, x_2)^\top$. To get rid of the difficulty due to the time-dependence of the domain we take the reference frame by making change of variables for $t \geq 0$ and $x \in \Omega$

$$\begin{aligned}y &= O(at)x, \quad u(x, t) = O(at)^\top v(y, t), \quad p(x, t) = q(y, t), \\ f(x, t) &= O(at)^\top g(y, t).\end{aligned}$$

Then (2) is equivalent to the equations:

$$\begin{cases} \partial_t u - \Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p = -u \cdot \nabla u + f, & t > 0, x \in \Omega, \\ \operatorname{div} u = 0, & t > 0, x \in \Omega. \end{cases}\tag{3}$$

In order to understand the structure of solutions at spatial infinity it is important to study this system in \mathbb{R}^2 . The effect of the boundary is expressed as a force in this case. Motivated

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by this observation, as a model problem, in this paper we study the above nonlinear system in \mathbb{R}^2 and in the steady case. Thus, assuming that f is independent of t , we are interested in the following system:

$$\begin{cases} -\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p = -u \cdot \nabla u + f, & x \in \mathbb{R}^2, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^2. \end{cases} \quad (\text{NS}_a)$$

Our aim is to show the existence and the asymptotic behavior of the solution to (NS_a) . For this purpose we first consider the linearized problem

$$\begin{cases} -\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p = f, & x \in \mathbb{R}^2, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^2. \end{cases} \quad (\text{S}_a)$$

We will show that there exists a unique solution to (S_a) such that the leading term of the asymptotic behavior of the flow at infinity is the rotational profile $c \frac{x^\perp}{|x|^2}$ whose coefficient c is determined by the external force f .

Before stating the main theorem, let us recall some known results on the mathematical analysis of flows around a rotating obstacle.

So far the mathematical results on this topic have been obtained mainly for the three-dimensional problem, as listed below. For the nonstationary problem the existence of global weak solutions is proved by Borchers [12], and the unique existence of time-local regular solutions is shown by Hishida [7] and Geissert, Heck, and Hieber [13], while the global strong solutions for small data are obtained by Galdi and Silvestre [14]. The spectrum of the linear operator related to this problem is studied by Farwig and Neustupa [1]; see also the linear analysis by Hishida [8]. The existence of stationary solutions to the associated system is proved in [12], Silvestre [15], Galdi [6], and Farwig and Hishida [2]. In particular, in [6] the stationary flows with the decay order $O(|x|^{-1})$ are obtained, while the work of [2] is based on the weak L^3 framework, which is another natural scale-critical space for the three-dimensional Navier-Stokes equations. In 3D case the asymptotic profiles of these stationary flows at spatial infinity are studied by Farwig and Hishida [3, 4] and Farwig, Galdi, and Kyed [5], where it is proved that the asymptotic profiles are described by the Landau solutions, stationary self-similar solutions to the Navier-Stokes equations in $\mathbb{R}^3 \setminus \{0\}$. It is worthwhile to mention that, also in the two-dimensional case, the asymptotic profile is given by the stationary self-similar solution $c \frac{x^\perp}{|x|^2}$. The stability of the above stationary solutions has been well studied in the three-dimensional case; The global L^2 stability is proved in [14], and the local L^3 stability is obtained by Hishida and Shibata [11].

All results mentioned above are considered in the three-dimensional case, while only a few results are known so far for the flow around a rotating obstacle in the two-dimensional case. An important progress has been made by Hishida [9], where the asymptotic behavior of the two-dimensional stationary Stokes flow around a rotating obstacle is investigated in details. Recently, the nonlinear problem (2) is analyzed in [10], and the existence of the unique solution decaying as $O(|x|^{-1})$ is proved for sufficiently small a and f when the external force f is of divergence form $f = \operatorname{div} F$ and F has a scale critical decay. Moreover, the leading profile at spatial infinity is shown as $C \frac{x^\perp}{|x|^2}$ under the additional decay condition on F such as $F = O(|x|^{-2-r})$, $r > 0$.

Since we consider the problem in \mathbb{R}^2 in this paper, by virtue of the absence of the physical boundary, we can show the existence of solutions to (NS_a) without assuming the

smallness of the angular velocity a . To state our result let us introduce the function space. For a fixed number $s \geq 0$ the weighted L^∞ space $L_s^\infty(\mathbb{R}^2)$ is defined as

$$L_s^\infty(\mathbb{R}^2) = \{f \in L^\infty(\mathbb{R}^2) \mid (1 + |x|)^s f \in L^\infty(\mathbb{R}^2)\}.$$

The space is a Banach space equipped with the natural norm

$$\|f\|_{L_s^\infty} = \text{ess.sup}_{x \in \mathbb{R}^2} (1 + |x|)^s |f(x)|.$$

The first result of this paper is stated as follows.

Theorem 1.1 *Let $a \in \mathbb{R} \setminus \{0\}$ and $r \in [0, 1)$. Assume that $f \in L_{3+r}^\infty(\mathbb{R}^2)^2$. Then there exists a unique $(u, p) \in L_1^\infty(\mathbb{R}^2)^2 \times L^\infty(\mathbb{R}^2)$ such that:*

1. *The couple (u, p) satisfies (S_a) in the sense of distributions.*
2. *The velocity u belongs to $L_1^\infty(\mathbb{R}^2)^2$ and satisfies*

$$u(x) = \int_{|y| < \frac{|x|}{2}} y^\perp \cdot f(y) \, dy \frac{x^\perp}{4\pi|x|^2} + \mathcal{R}[f](x), \quad (4)$$

with

$$|\mathcal{R}[f](x)| \leq \frac{C}{(1 + |x|)^{1+r}} \left(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}} \right) \|f\|_{L_{3+r}^\infty}, \quad (5)$$

and in particular, it follows that

$$\|\mathcal{R}[f]\|_{L_{1+r}^\infty} \leq C \left(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}} \right) \|f\|_{L_{3+r}^\infty} \quad (6)$$

with a numerical constant C .

3. *The pressure p is given by*

$$p(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \cdot f(y) \, dy. \quad (7)$$

Remark 1.2 (1) The representation (7) leads to the regularity of the pressure such as $p \in L_1^\infty(\mathbb{R}^2)$. The solution $(u, p) \in L_1^\infty(\mathbb{R}^2)^2 \times L_1^\infty(\mathbb{R}^2)$ satisfying (S_a) in the sense of distributions is unique by virtue of the uniqueness result in Hishida [9, Lemma 3.5].

(2) In [9] the result of Theorem 1.1 is firstly established under the conditions on f such as $f \in L^1(\mathbb{R}^2)^2 \cap L^\infty(\mathbb{R}^2)^2$, $x^\perp \cdot f \in L^1(\mathbb{R}^2)^2$, and $f(x) = O(|x|^{-3}(\log |x|)^{-1})$ as $|x| \rightarrow \infty$. Our result improves his result, and in particular, the critical case $f = O(|x|^{-3})$ is treated. We note that, in the case $r = 0$, the integral in the right-hand side of (4) does not converge in general when $|x| \rightarrow \infty$.

To study the nonlinear problem it is reasonable to consider the linear problem when the external force f is given by $f = \text{div } F$ with $F(x) = O(|x|^{-2})$ in view of the structure of the nonlinear term $u \cdot \nabla u = \text{div } (u \otimes u)$. Here the matrix $(u_i v_j)_{1 \leq i, j \leq 2}$ is written as $u \otimes v$. The following result is essentially obtained in [10].

Theorem 1.3 ([10, Theorem 3.1(ii)]) *Let $a \in \mathbb{R} \setminus \{0\}$, $r \in [0, 1)$, and $q \in (1, \infty)$. Assume that $f \in L^2(\mathbb{R}^2)^2$ is of divergence form $f = \operatorname{div} F = (\partial_1 F_{11} + \partial_2 F_{12}, \partial_1 F_{21} + \partial_2 F_{22})^\top$ with $F = (F_{ij})_{1 \leq i, j \leq 2} \in L_{2+r}^\infty(\mathbb{R}^2)^{2 \times 2}$. Then there exists a unique $(u, p) \in L_1^\infty(\mathbb{R}^2)^2 \times L^q(\mathbb{R}^2)$ such that (u, p) satisfies (S_a) in the sense of distributions, and u satisfies*

$$u(x) = \int_{|y| < \frac{|x|}{2}} (F_{12}(y) - F_{21}(y)) dy \frac{x^\perp}{4\pi|x|^2} + \mathcal{R}[f](x), \quad (8)$$

and $\mathcal{R}[f]$ satisfies

$$\begin{aligned} |\mathcal{R}[f](x)| &\leq C \min \left\{ \frac{1}{|a||x|^3}, \frac{1}{|x|} \right\} \int_{|y| \leq \frac{|x|}{2}} |F(y)| dy \\ &\quad + \frac{C}{(1+|x|)^{1+r}} \frac{1}{1-r} \|F\|_{L_{2+r}^\infty}. \end{aligned} \quad (9)$$

Here C is a numerical constant independent also of a and r . In particular, it follows that

$$\|\mathcal{R}[f]\|_{L_{1+r}^\infty} \leq C \left(\frac{1}{1-r} + \frac{1 + \log |a|}{|a|^{\frac{r}{2}}} \right) \|F\|_{L_{2+r}^\infty} \quad (10)$$

with a numerical constant C .

Remark 1.4 (1) In fact, the statement of [10, Theorem 3.1] is a slightly different from Theorem 1.3 above. So we give a sketch of the proof of Theorem 1.3 in Section 2, based on the key pointwise asymptotic estimate of the fundamental solution, see Lemma 2.2 below, which is due to [10, Lemma 3.3].

(2) Estimate (10) is derived from (9). Indeed, when $|x| \leq 1$ the first term in the right-hand side of (9) is estimated as $C|x|\|F\|_{L^\infty}$, while when $|x| \geq 1$ this term is estimated by dividing into two cases (i) $|a||x|^2 \leq 1$ and (ii) $|a||x|^2 \geq 1$. The factor $\log |a|$ in (10) is required only when r is near 0 in order to ensure that the constant C is independent of r .

The linear results of Theorem 1.1 and 1.3 are applied to the nonlinear problem (NS_a) . The result for the nonlinear problem is stated as follows.

Theorem 1.5 *Let $a \in \mathbb{R} \setminus \{0\}$ and $r \in [0, 1)$. Then there exists $\delta = \delta(a, r) > 0$ such that, for any $f \in L_{3+r}^\infty(\mathbb{R}^2)^2$ satisfying $x^\perp \cdot f \in L^1(\mathbb{R}^2)$ and*

$$\|x^\perp \cdot f\|_{L^1} + \|f\|_{L_{3+r}^\infty} < \delta, \quad (11)$$

there exists a unique solution (u, p) to (NS_a) such that

$$u(x) = \alpha U(x) + v(x), \quad (12)$$

where

$$\alpha = \frac{1}{2} \int_{\mathbb{R}^2} y^\perp \cdot f(y) dy, \quad U(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-\frac{|x|^2}{4}}), \quad (13)$$

and

$$\|v\|_{L_{1+r}^\infty} \leq C_r \left(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}} \right) (\|x^\perp \cdot f\|_{L^1} + \|f\|_{L_{3+r}^\infty}), \quad (14)$$

and the pressure p is given by

$$p(x) = \nabla \cdot (-\Delta)^{-1} \nabla \cdot (u \otimes u) - \nabla \cdot (-\Delta)^{-1} f. \quad (15)$$

Here the constant C_r depends only on r .

Remark 1.6 In Theorem 1.5 the solution is constructed as the solution to the integral equation associated with (NS_a) , which is formulated based on the fundamental solution to the linearized problem (S_a) . The uniqueness is proved for this class of solutions.

This paper is organized as follows. In Section 2 we collect the estimates which reflect the effect of the rotation. Most of them are the abstractions from [9, 10]. Theorem 1.1 is proved in Section 3. Finally, Theorem 1.5 is proved in Section 4.

2 Preliminaries

Let us consider the linear problem in the whole plane for $a \in \mathbb{R} \setminus \{0\}$:

$$-\Delta u - a(x^\perp \cdot \nabla u - u^\perp) + \nabla p = f, \quad \operatorname{div} u = 0, \quad x \in \mathbb{R}^2. \quad (S_a)$$

Let $q \in [1, \infty]$. The couple $(u, p) \in L^\infty(\mathbb{R}^2)^2 \times L^q(\mathbb{R}^2)$ is said to be a weak solution to (S_a) if (i) $\operatorname{div} u = 0$ in the sense of distributions, and (ii) (u, p) satisfies

$$\int_{\mathbb{R}^2} u \cdot T_{-a} \phi \, dx - \int_{\mathbb{R}^2} p \operatorname{div} \phi \, dx = \int_{\mathbb{R}^2} f \cdot \phi \, dx, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^2)^2,$$

where

$$T_a u = -\Delta u - a(x^\perp \cdot \nabla u - u^\perp).$$

Let $\mathbb{I} = (\delta_{ij})_{1 \leq i, j \leq 2}$ be the identity matrix. The velocity part of the fundamental solution to (S_a) plays a central role throughout this paper, which is defined as

$$\Gamma_a(x, y) = \int_0^\infty O(at)^\top K(O(at)x - y, t) \, dt, \quad (16)$$

where

$$K(x, t) = G(x, t)\mathbb{I} + H(x, t), \quad H(x, t) = \int_t^\infty \nabla^2 G(x, s) \, ds, \quad (17)$$

and $G(x, t)$ is the two-dimensional Gauss kernel

$$G(x, t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.$$

Similarly, the pressure part of the fundamental solution is defined as

$$Q(x - y) = \frac{1}{2\pi} \log |x - y|,$$

for the following identity holds.

$$\operatorname{div}(x^\perp \cdot \nabla u - u^\perp) = x^\perp \cdot \nabla \operatorname{div} u = 0.$$

Remark 2.1 We can also write $H(x, t)$ in (17) as follows

$$H(x, t) = -\frac{(x \otimes x)}{|x|^2} G(x, t) + \left(\frac{x \otimes x}{|x|^2} - \frac{\mathbb{I}}{2} \right) \frac{1 - e^{-\frac{|x|^2}{4t}}}{\pi |x|^2}.$$

The next lemma is proved in [9, 10].

Lemma 2.2 ([9, Proposition 3.1], [10, Lemma 3.3]) *Set*

$$L(x, y) = \frac{x^\perp \otimes y^\perp}{4\pi|x|^2}. \quad (18)$$

Then for $m = 0, 1$ the kernel $\Gamma_a(x, y)$ satisfies

$$\begin{aligned} & |\nabla_y^m (\Gamma_a(x, y) - L(x, y))| \\ & \leq C \left(\delta_{0m} \min \left\{ \frac{1}{|a||x|^2}, \frac{1}{|a|^{\frac{1}{2}}|x|} \right\} + |x|^{1-m} \min \left\{ \frac{1}{|a||x|^3}, \frac{1}{|x|} \right\} + \frac{|y|^{2-m}}{|x|^2} \right), \quad (19) \\ & \text{for } |x| > 2|y|. \end{aligned}$$

Here δ_{0m} is the Kronecker delta and C is independent of x, y , and a .

Remark 2.3 (1) The asymptotic estimate like (19) is proved in [9] when $m = 0$, and then the dependence on $|a|$ is improved by [10] which is needed to solve the nonlinear problem. The detailed proof for the case $m = 1$ of (19) is given by [10].

(2) Note that, when $|y| > 2|x|$, since $\Gamma_a(x, y) = \Gamma_{-a}(y, x)^\top$ and $(y^\perp \otimes x^\perp)^\top = x^\perp \otimes y^\perp$ we have a similar estimate:

$$\begin{aligned} & \left| \Gamma_a(x, y) - \frac{x^\perp \otimes y^\perp}{4\pi|y|^2} \right| \\ & \leq C \left(\min \left\{ \frac{1}{|a||y|^2}, \frac{1}{|a|^{\frac{1}{2}}|y|} \right\} + |y| \min \left\{ \frac{1}{|a||y|^3}, \frac{1}{|y|} \right\} + \frac{|x|^2}{|y|^2} \right), \quad (20) \\ & \text{for } |y| > 2|x|. \end{aligned}$$

Proof of Theorem 1.3. Here we give a sketch of the proof of Theorem 1.3. The unique solution u to (S_a) decaying at spatial infinity is expressed as $u(x) = \int_{\mathbb{R}^2} \Gamma_a(x, y) f(y) dy$, and we focus on the proof of (9) and (10). By the integration by parts we have

$$\begin{aligned} \int_{\mathbb{R}^2} \Gamma_a(x, y) f(y) dy &= - \int_{\mathbb{R}^2} \nabla_y \Gamma_a(x, y) F(y) dy \\ &= - \left(\int_{|y| < \frac{|x|}{2}} + \int_{\frac{|x|}{2} \leq |y|} \right) \nabla_y \Gamma_a(x, y) F(y) dy \\ &= I(x) + II(x). \end{aligned}$$

The term I is further decomposed as

$$\begin{aligned} I(x) &= - \int_{|y| < \frac{|x|}{2}} \nabla_y L(x, y) F(y) dy - \int_{|y| < \frac{|x|}{2}} \nabla_y (\Gamma_a(x, y) - L(x, y)) F(y) dy \\ &= I_1(x) + I_2(x). \end{aligned}$$

By the definition of $L(x, y)$ we have $-(\nabla_y L(x, y)) F = (F_{12} - F_{21}) \frac{x^\perp}{4\pi|x|^2}$, which implies

$$I_1(x) = \int_{|y| < \frac{|x|}{2}} (F_{12}(y) - F_{21}(y)) dy \frac{x^\perp}{4\pi|x|^2}. \quad (21)$$

As for I_2 , when $|x| \geq 1$ we have from (19) with $m = 1$,

$$\begin{aligned} |I_2(x)| &\leq C \min \left\{ \frac{1}{|a||x|^3}, \frac{1}{|x|} \right\} \int_{|y| \leq \frac{|x|}{2}} |F(y)| \, dy + \frac{C}{|x|^2} \int_{|y| \leq \frac{|x|}{2}} |y| |F(y)| \, dy \\ &\leq C \min \left\{ \frac{1}{|a||x|^3}, \frac{1}{|x|} \right\} \int_{|y| \leq \frac{|x|}{2}} |F(y)| \, dy + \frac{C}{(1+|x|)^{1+r}} \frac{1}{1-r} \|F\|_{L_{2+r}^\infty}, \end{aligned} \quad (22)$$

where C is a numerical constant independent also of r . Next we have from the direct calculation

$$|(\nabla_x K)(x, t)| \leq C \left(t^{-\frac{3}{2}} e^{-\frac{|x|^2}{16t}} + \int_t^\infty s^{-\frac{5}{2}} e^{-\frac{|x|^2}{16s}} \, ds \right),$$

which implies

$$\int_0^\infty |(\nabla K)(O(at)x, t)| \, dt \leq \frac{C}{|x|}, \quad x \neq 0.$$

Then by the change of the variables $y = O(at)z$ we have

$$\begin{aligned} |II(x)| &\leq \left| \int_{|y| \geq \frac{|x|}{2}} \nabla_y \Gamma_a(x, y) F(y) \, dy \right| \\ &\leq \int_0^\infty \int_{|y| \geq \frac{|x|}{2}} |(\nabla K)(O(at)x - y, t)| |F(y)| \, dy \, dt \\ &\leq C \|F\|_{L_{2+r}^\infty} \int_{|z| \geq \frac{|x|}{2}} \left(\int_0^\infty |(\nabla K)(O(at)(x - z), t)| \, dt \right) (1 + |z|)^{-2-\gamma} \, dz \\ &\leq C \|F\|_{L_{2+r}^\infty} \int_{|z| \geq \frac{|x|}{2}} |x - z|^{-1} (1 + |z|)^{-2-\gamma} \, dz \\ &\leq \frac{C}{(1 + |x|)^{1+\gamma}} \|F\|_{L_{2+r}^\infty}. \end{aligned} \quad (23)$$

Here C is a numerical constant. From (21), (22), and (23), we conclude (8) and (9) for $r \in [0, 1)$. The proof of Theorem 1.3 is complete.

3 Proof of linear result

In this section we prove Theorem 1.1. Set

$$L[f](x) = \int_{\mathbb{R}^2} \Gamma_a(x, y) f(y) \, dy, \quad (24)$$

where $\Gamma_a(x, y)$ is given by (16). It is known by [9, Lemma 3.5] that $u = L[f]$ together with p defined by (7) is the unique weak solution to (S_a) decaying at spatial infinity. So we focus on the proof of the estimates (4) and (5) here. To apply Lemma 2.2 we first divide (24) into three parts:

$$\begin{aligned} L[f](x) &= U_1(x) + U_2(x) + U_3(x) \\ &= \left(\int_{|y| < \frac{|x|}{2}} + \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} + \int_{2|x| < |y|} \right) \Gamma_a(x, y) f(y) \, dy. \end{aligned}$$

By Lemma 2.2 we have

$$U_1(x) = \int_{|y| < \frac{|x|}{2}} y^\perp \cdot f(y) \, dy \frac{x^\perp}{4\pi|x|^2} + W_1(x) \quad (25)$$

with

$$\begin{aligned} |W_1(x)| &\leq C \min \left\{ \frac{1}{|a||x|^2}, \frac{1}{|a|^{\frac{1}{2}}|x|} \right\} \int_{|y| < \frac{|x|}{2}} |f(y)| \, dy \\ &\quad + C \min \left\{ \frac{1}{|a||x|^2}, 1 \right\} \int_{|y| < \frac{|x|}{2}} |f(y)| \, dy + \frac{C}{|x|^2} \int_{|y| < \frac{|x|}{2}} |y|^2 |f(y)| \, dy \\ &\leq \begin{cases} C \left(\frac{1}{|a|^{\frac{1}{2}}} + 1 \right) \|f\|_{L^\infty}, & |x| \leq 1, \\ C \left\{ \frac{1}{|a|^{\frac{1+r}{2}}|x|^{1+r}} + \frac{1}{(1-r)|x|^{1+r}} \right\} \|f\|_{L_{3+r}^\infty}, & |x| \geq 1 \end{cases} \\ &\leq \frac{C}{(1+|x|)^{1+r}} \left(\frac{1}{|a|^{\frac{1+r}{2}}} + \frac{1}{1-r} \right) \|f\|_{L_{3+r}^\infty}. \end{aligned} \quad (26)$$

Similarly, by Remark 2.3 we have

$$\begin{aligned} |U_3(x)| &\leq C \int_{2|x| < |y|} \left(\min \left\{ \frac{1}{|a||y|^2}, \frac{1}{|a|^{\frac{1}{2}}|y|} \right\} + \min \left\{ \frac{1}{|a||y|^2}, 1 \right\} + \frac{|x|}{|y|} \right) |f(y)| \, dy \\ &\leq \begin{cases} C \left(\frac{1}{|a|^{\frac{1}{2}}} + 1 \right) \|f\|_{L_{3+r}^\infty}, & |x| \leq 1, \\ \frac{C}{|x|^{1+r}} \left(\frac{1}{|a|^{\frac{1}{2}}} + 1 \right) \|f\|_{L_{3+r}^\infty}, & |x| \geq 1. \end{cases} \end{aligned} \quad (27)$$

From (27) we have

$$|U_3(x)| \leq \frac{C}{(1+|x|)^{1+r}} \left(\frac{1}{|a|^{\frac{1}{2}}} + 1 \right) \|f\|_{L_{3+r}^\infty} \quad (28)$$

with a numerical constant C . Finally, we decompose $U_2(x)$ as

$$U_2(x) = U_{2,1}(x) + U_{2,2}(x), \quad U_{2,1} = U_{2,1,1} + U_{2,1,2} \quad (29)$$

with

$$\begin{aligned} U_{2,1,1}(x) &= \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_0^l O(at)^\top \frac{1}{8\pi t} e^{-\frac{|O(at)x-y|^2}{4t}} f(y) \, dt \, dy, \\ U_{2,1,2}(x) &= \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_l^\infty O(at)^\top \frac{1}{8\pi t} e^{-\frac{|O(at)x-y|^2}{4t}} f(y) \, dt \, dy, \\ U_{2,2}(x) &= \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_0^\infty O(at)^\top \left(K(O(at)x-y) - \frac{1}{8\pi t} e^{-\frac{|O(at)x-y|^2}{4t}} \mathbb{I} \right) f(y) \, dt \, dy, \end{aligned}$$

where $l = l(a, |x|) > 0$ will be chosen later. We start from the estimate of $U_{2,1,1}(x)$. By Fubini's theorem and changing the variable as $z = O(at)x - y$ we obtain

$$\begin{aligned}
|U_{2,1,1}(x)| &\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_0^l t^{-1} e^{-\frac{|O(at)x-y|^2}{4t}} dt dy \\
&\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \int_0^l \int_{\mathbb{R}^2} t^{-1} e^{-\frac{|O(at)x-y|^2}{4t}} dy dt \\
&\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \int_0^l \int_{\mathbb{R}^2} t^{-1} e^{-\frac{|z|^2}{4t}} dz dt \\
&\leq \frac{C}{(1+|x|)^{3+r}} l \|f\|_{L_{3+r}^\infty}.
\end{aligned} \tag{30}$$

Here C is a numerical constant. Next we estimate $U_{2,1,2}$. Since

$$O(at)^\top = -\frac{1}{a} \frac{d}{dt} \dot{O}(at)^\top,$$

the integrating by parts yields

$$\begin{aligned}
U_{2,1,2}(x) &= -\frac{1}{2a} \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_l^\infty \left(\frac{d}{dt} \dot{O}(at)^\top \right) G(O(at)x - y, t) f(y) dt dy \\
&= \frac{1}{2a} \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_l^\infty \dot{O}(at)^\top \frac{d}{dt} (G(O(at)x - y, t)) f(y) dt dy + W_2(x),
\end{aligned} \tag{31}$$

and the remainder term W_2 is estimated as

$$\begin{aligned}
|W_2(x)| &\leq \frac{C}{|a|} \int_{\frac{|x|}{2} \leq |y| \leq |x|} |G(O(al)x - y, l) f(y)| dy \\
&\leq \frac{C}{(1+|x|)^{1+r}} \frac{1}{l|a|} \|f\|_{L_{3+r}^\infty}.
\end{aligned} \tag{32}$$

To estimate the first term in the right-hand side of (31) we use the following calculation,

$$\begin{aligned}
&\frac{d}{dt} G(O(at)x - y, t) \\
&= \frac{e^{-\frac{|O(at)x-y|^2}{4t}}}{4\pi} \left\{ -t^2 + t^{-3} \frac{|O(at)x - y|^2}{4} - at^{-2} \frac{(\dot{O}(at)x) \cdot (O(at)x - y)}{2} \right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\left| \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_l^\infty \dot{O}(at)^\top \frac{d}{dt} (G(O(at)x - y, t)) f(y) dt dy \right| \\
&\leq C \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_l^\infty (t^{-2} + t^{-3} |O(at)x - y|^2) e^{-\frac{|O(at)x-y|^2}{4t}} |f(y)| dt dy \\
&\quad + C |a| \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_l^\infty t^{-2} |x| |O(at)x - y| e^{-\frac{|O(at)x-y|^2}{4t}} |f(y)| dt dy \\
&\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \left(\frac{1}{l} |x|^2 + |a||x| \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{|O(at)x-y|^2}{8t}} dt dy \right),
\end{aligned}$$

and then, by the change of variables as $z = O(at)^\top y$ we see

$$\begin{aligned}
&\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \left(\frac{1}{l} |x|^2 + |a||x| \int_{\frac{|x|}{2} \leq |z| \leq 2|x|} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{|x-z|^2}{8t}} dt dz \right) \\
&\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \left(\frac{1}{l} |x|^2 + |a||x| \int_{|x-z| \leq 3|x|} \frac{dz}{|x-z|} \right) \\
&\leq \frac{C}{(1+|x|)^{1+r}} \left(\frac{1}{l} + |a| \right) \|f\|_{L_{3+r}^\infty}.
\end{aligned} \tag{33}$$

Then (31), (32), and (33) implies that

$$|U_{2,1,2}(x)| \leq \frac{C}{(1+|x|)^{1+r}} \left(\frac{1}{l|a|} + 1 \right) \|f\|_{L_{3+r}^\infty}. \tag{34}$$

Here C is a numerical constant. On the other hand, the term $U_{2,2}$ converges absolutely without using the effect of rotation. Indeed, changing the variables $y = O(at)z$, we have

$$\begin{aligned}
&\left| \int_{\frac{|x|}{2} \leq |y| \leq 2|x|} \int_0^\infty O(at)^\top \left(K(O(at)x - y, t) - \frac{1}{8\pi t} e^{-\frac{|O(at)x - y|^2}{4t}} \mathbb{I} \right) f(y) dt dy \right| \\
&\leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty} \int_{|z| \leq 2|x|} \int_0^\infty |B(x - z, t)| dt dz,
\end{aligned} \tag{35}$$

where $B(x, t)$ is given by

$$B(x, t) = \left(\frac{e^{-\frac{|x|^2}{4t}}}{8\pi t} - \frac{1 - e^{-\frac{|x|^2}{4t}}}{2\pi|x|^2} \right) \left(\mathbb{I} - 2 \frac{x \otimes x}{|x|^2} \right).$$

For any fixed $x - z$ we have from the change of variables as $s = \frac{|x-z|^2}{4t}$,

$$\int_0^\infty |B(x - z, t)| dt \leq \left| \mathbb{I} - 2 \frac{(x - z) \otimes (x - z)}{|x - z|^2} \right| \int_0^\infty \frac{-se^{-s} + 1 - e^{-s}}{s^2} ds \leq C, \tag{36}$$

where C is independent of $x - z$. Here we have used the identity

$$\frac{d}{ds} \left(\frac{e^{-s} - 1}{s} \right) = \frac{-se^{-s} + 1 - e^{-s}}{s^2} > 0.$$

We combine (35) with (36) to conclude

$$|U_{2,2}(x)| \leq \frac{C}{(1+|x|)^{3+r}} \|f\|_{L_{3+r}^\infty}. \tag{37}$$

Here C is a numerical constant. Collecting (29), (30), (34), and (37), we see

$$|U_2(x)| \leq \frac{C}{(1+|x|)^{1+r}} \left\{ \frac{l}{(1+|x|)^2} + \frac{1}{l|a|} + 1 \right\} \|f\|_{L_{3+r}^\infty},$$

and thus, by taking $l = \frac{1+|x|}{|a|^{\frac{1}{2}}}$,

$$|U_2(x)| \leq \frac{C}{(1+|x|)^{1+r}} \left\{ \frac{1}{(1+|x|)|a|^{\frac{1}{2}}} + 1 \right\} \|f\|_{L_{3+r}^\infty}. \tag{38}$$

From (24), (25), (26), (28), and (38), we obtain (4) and (5). The proof of Theorem 1.1 is complete.

4 Proof of nonlinear result

We are now in a position to give a proof of our main result. The unique existence and the asymptotic behavior of solutions to (NS_a) will be obtained by combining the results of Theorem 1.3 and Theorem 1.1 by applying the standard fixed point argument. For $r \in [0, 1)$ and $\delta \in (0, 1)$ we introduce the function space $X_{r,\delta}$ as follows.

$$X_{r,\delta} = \{v \in L_{1+r}^\infty(\mathbb{R}^2)^2 \mid \|v\|_{L_{1+r}^\infty} \leq \delta, \operatorname{div} v = 0\}.$$

We also set

$$\begin{aligned} U(x) &= \frac{1}{2\pi} \frac{x^\perp}{|x|^2} (1 - e^{-\frac{|x|^2}{4}}), \\ \alpha &= \frac{\int_{\mathbb{R}^2} y^\perp \cdot f(y) \, dy}{\int_{\mathbb{R}^2} y^\perp \cdot \Delta U(y) \, dy} = \frac{1}{2} \int_{\mathbb{R}^2} y^\perp \cdot f(y) \, dy, \\ w(x) &= u(x) - \alpha U(x), \end{aligned} \tag{39}$$

Here we have used the fact $\int_{\mathbb{R}^2} x^\perp \cdot \Delta U \, dx = 2$, which is derived from the identity

$$\Delta U = (-\partial_2 G, \partial_1 G)^\top, \quad G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}}.$$

The direct computation leads to the existence of a scalar function $P_U \in L^\infty(\mathbb{R}^2)$ such that

$$-a(x^\perp \cdot \nabla \alpha U - \alpha U^\perp) + \alpha^2 U \cdot \nabla U = \nabla P_U.$$

Then w satisfies the following equations in \mathbb{R}^2 :

$$\begin{cases} -\Delta w - a(x^\perp \cdot \nabla w - w^\perp) + \nabla \pi \\ \quad = -\alpha(U \cdot \nabla w + w \cdot \nabla U) - w \cdot \nabla w - \alpha \Delta U + f, \\ \operatorname{div} w = 0. \end{cases}$$

Here $\pi = p - P_U$. Let us recall that for $f \in L_3^\infty(\mathbb{R}^2)^2$, the function $L[f](x) = \int_{\mathbb{R}^2} \Gamma_a(x, y) f(y) \, dy$ defines the unique weak solution to (S_a) decaying at spatial infinity. Then we introduce the map Φ as

$$\Phi[w](x) = L[-\alpha(U \cdot \nabla w + w \cdot \nabla U) - w \cdot \nabla w - \alpha \Delta U + f](x). \tag{40}$$

Here we consider the leading profile of (40). Since $U \cdot \nabla w + w \cdot \nabla U = \operatorname{div}(U \otimes w + w \otimes U)$ and $w \cdot \nabla w = \operatorname{div}(w \otimes w)$, we see that

$$\begin{aligned} & \frac{1}{4\pi} \left\{ \int_{|y| < \frac{|x|}{2}} (U \otimes w + w \otimes U)_{2,1} - (U \otimes w + w \otimes U)_{1,2} \, dy \right. \\ & \quad \left. + \int_{|y| < \frac{|x|}{2}} (w \otimes w)_{2,1} - (w \otimes w)_{1,2} \, dy \right\} \\ &= 0. \end{aligned} \tag{41}$$

From (40) and (41), Theorems 1.1 and 1.3 yield

$$\begin{aligned} \Phi[w](x) &= \mathcal{R}[-\alpha(U \cdot \nabla w + w \cdot \nabla U) - w \cdot \nabla w - \alpha \Delta U + f](x) \\ & \quad + \left(\int_{|y| < \frac{|x|}{2}} y^\perp \cdot f \, dy - \alpha \int_{|y| < \frac{|x|}{2}} y^\perp \cdot \Delta U \, dy \right) \frac{x^\perp}{4\pi|x|^2}. \end{aligned} \tag{42}$$

To estimate the last term of (42), we have from the definition of α in (39) and $\int_{\mathbb{R}^2} x^\perp \cdot \Delta U \, dx = 2$,

$$\begin{aligned}
& \left| \int_{|y| < \frac{|x|}{2}} y^\perp \cdot f \, dy - \alpha \int_{|y| < \frac{|x|}{2}} y^\perp \cdot \Delta U \, dy \right| \\
&= \frac{1}{2} \left| \int_{\mathbb{R}^2} y^\perp \cdot \Delta U \, dy \int_{|y| < \frac{|x|}{2}} y^\perp \cdot f \, dy - \int_{|y| < \frac{|x|}{2}} y^\perp \cdot \Delta U \, dy \int_{\mathbb{R}^2} y^\perp \cdot f \, dy \right| \\
&= \frac{1}{2} \left| \int_{\mathbb{R}^2} y^\perp \cdot \Delta U \, dy \left(\int_{|y| < \frac{|x|}{2}} y^\perp \cdot f \, dy - \int_{\mathbb{R}^2} y^\perp \cdot f \, dy \right) \right. \\
&\quad \left. - \left(\int_{|y| < \frac{|x|}{2}} y^\perp \cdot \Delta U \, dy - \int_{\mathbb{R}^2} y^\perp \cdot \Delta U \, dy \right) \int_{\mathbb{R}^2} y^\perp \cdot f \, dy \right| \\
&= \frac{1}{2} \left| - \int_{\mathbb{R}^2} y^\perp \cdot \Delta U \, dy \int_{\frac{|x|}{2} \leq |y|} y^\perp \cdot f \, dy + \int_{\frac{|x|}{2} \leq |y|} y^\perp \cdot \Delta U \, dy \int_{\mathbb{R}^2} y^\perp \cdot f \, dy \right| \\
&\leq \frac{C_r}{(1+|x|)^r} (\|y^\perp \cdot f\|_{L^1} + \|f\|_{L_{3+r}^\infty}), \quad \text{for } |x| > 1. \tag{43}
\end{aligned}$$

Here the constant C_r depends only on r . Note that the boundedness of $\|y^\perp \cdot f\|_{L^1}$ is always valid when $r > 0$. When $|x| \leq 1$ it is easy to see

$$\left| \int_{|y| < \frac{|x|}{2}} y^\perp \cdot f \, dy - \alpha \int_{|y| < \frac{|x|}{2}} y^\perp \cdot \Delta U \, dy \right| \leq C(\|y^\perp \cdot f\|_{L^1} + \|f\|_{L^\infty}). \tag{44}$$

Estimates (43) and (44) combined with (42) imply that

$$\begin{aligned}
\Phi[w](x) &= \mathcal{R}[-\alpha(U \cdot \nabla w + w \cdot \nabla U) - w \cdot \nabla w - \alpha \Delta U + f](x) \\
&\quad + O\left((\|y^\perp \cdot f\|_{L^1} + \|f\|_{L_{3+r}^\infty})(1+|x|)^{-1-r}\right). \tag{45}
\end{aligned}$$

By (45) and Theorems 1.1 and 1.3, we can verify that

$$\begin{aligned}
\|\Phi[w]\|_{L_{1+r}^\infty} &\leq C\left(\frac{1}{1-r} + \frac{1+\log|a|}{|a|^{\frac{r}{2}}}\right) \|\alpha(U \otimes w + w \otimes U) + \frac{1}{2}w \otimes w\|_{L_{2+r}^\infty} \\
&\quad + C\left(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}}\right) \|\alpha \Delta U + f\|_{L_{3+r}^\infty} \\
&\quad + C_r(\|y^\perp \cdot f\|_{L^1} + \|f\|_{L_{3+r}^\infty}) \\
&\leq C\left(\frac{1}{1-r} + \frac{1+\log|a|}{|a|^{\frac{r}{2}}}\right) (\|\alpha\| \|w\|_{L_{1+r}^\infty} + \|w\|_{L_{1+r}^\infty}^2) \\
&\quad + C_r\left(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}}\right) (\|f\|_{L_{3+r}^\infty} + \|y^\perp \cdot f\|_{L^1}), \tag{46}
\end{aligned}$$

where C_r depends only on r , while C is a numerical constant. We may take C and C_r larger than 1, and note that $|\alpha| \leq 2^{-1} \|y^\perp \cdot f\|_{L^1}$. If

$$0 < \delta \leq \frac{1}{3C\left(\frac{1}{1-r} + \frac{1+\log|a|}{|a|^{\frac{r}{2}}}\right)}$$

and

$$\lambda(f) = \|f\|_{L_{3+r}^\infty} + \|y^\perp \cdot f\|_{L^1} \leq \frac{\delta}{3C_r\left(\frac{1}{1-r} + \frac{1}{|a|^{\frac{1+r}{2}}}\right)},$$

then we see that $\Phi[w]$ becomes a mapping from $X_{r,\delta}$ into $X_{r,\delta}$. Moreover, from (10) there is a numerical constant $C' > 0$ such that

$$\begin{aligned}
& \|\Phi[w_1] - \Phi[w_2]\|_{L_{1+r}^\infty} \\
&= \|\mathcal{R}[-\alpha\{U \cdot \nabla(w_1 - w_2) + (w_1 - w_2) \cdot \nabla U\} - w_1 \cdot \nabla w_1 + w_2 \cdot \nabla w_2]\|_{L_{1+r}^\infty} \\
&\leq C' \left(\frac{1}{1-r} + \frac{1 + \log |a|}{|a|^{\frac{r}{2}}} \right) \|\alpha\{U \otimes (w_1 - w_2) + (w_1 - w_2) \otimes U\} \\
&\quad - \frac{1}{2}(w_1 \otimes w_1 - w_2 \otimes w_2)\|_{L_{2+r}^\infty} \\
&\leq C' \left(\frac{1}{1-r} + \frac{1 + \log |a|}{|a|^{\frac{r}{2}}} \right) (|\alpha| + \|w_1\|_{L_{1+r}^\infty} + \|w_2\|_{L_{1+r}^\infty}) \|w_1 - w_2\|_{L_{1+r}^\infty} \\
&\leq C' \left(\frac{1}{1-r} + \frac{1 + \log |a|}{|a|^{\frac{r}{2}}} \right) \left(\frac{\lambda(f)}{2} + 2\delta \right) \|w_1 - w_2\|_{L_{1+r}^\infty} \\
&\leq 3\delta C' \left(\frac{1}{1-r} + \frac{1 + \log |a|}{|a|^{\frac{r}{2}}} \right) \|w_1 - w_2\|_{L_{1+r}^\infty} \\
&= \tau \|w_1 - w_2\|_{L_{1+r}^\infty},
\end{aligned}$$

for all $w_1, w_2 \in X_{r,\delta}$, where we have set $\tau = 3\delta C' \left(\frac{1}{1-r} + \frac{1 + \log |a|}{|a|^{\frac{r}{2}}} \right)$. Hence, if δ (and thus, also $\lambda(f)$) is sufficiently small so that $\tau \in (0, 1)$ is justified, then we can conclude that Φ is a contraction on $X_{r,\delta}$. By the fixed point theorem, there exists a fixed point v , which is unique in $X_{r,\delta}$, such that

$$u(x) = \alpha U(x) + v(x), \quad v \in X_{r,\delta}$$

is a unique solution to (NS_a) with the pressure p defined by (15). Finally, the estimate (14) follows from (46) for the fixed point v of Φ by virtue of the smallness of $|\alpha|$ and δ . The proof of Theorem 1.5 is complete.

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